

ON RECTANGULAR DIAGRAMS, LEGENDRIAN KNOTS AND TRANSVERSE KNOTS

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1. INTRODUCTION

A correspondence is studied in [7] between front projections of Legendrian links in $(\mathbb{R}^3, \xi_{\text{std}})$ and rectangular diagrams. In this paper, we introduce braided rectangular diagrams, and study a relationship with Legendrian links in $(\mathbb{R}^3, \xi_{\text{sym}})$. We show Alexander and Markov Theorems for Legendrian links in $(\mathbb{R}^3, \xi_{\text{sym}})$.

We review a relationship between front projections of Legendrian links in $(\mathbb{R}^3, \xi_{\text{std}})$ and rectangular diagrams. The standard contact structure ξ_{std} in \mathbb{R}^3 is defined by $\ker(dz - ydx)$. It is well-known [12] that every Legendrian link in $(\mathbb{R}^3, \xi_{\text{std}})$ has a front projection with transverse double points and cusp singularities. Figure 1 (1) illustrates a front projection of a topologically trivial Legendrian knot. Changing every point with a horizontal tangent in a front projection to a corner, Figure 1 (2), followed by rotating the obtained diagram 45 degree clockwise, we obtain a rectangular diagram, Figure 1 (4).

The intersection of the plane $\{(x, y, z) \mid y = a\}$ with contact planes, called a *characteristic foliation*, consists of lines $z = ax + (\text{constant})$, where $a \in \mathbb{R}$. The characteristic foliation on the plane $\{(x, y, z) \mid x = b\}$ consists of lines $z = (\text{constant})$, where $b \in \mathbb{R}$. Changing every vertical

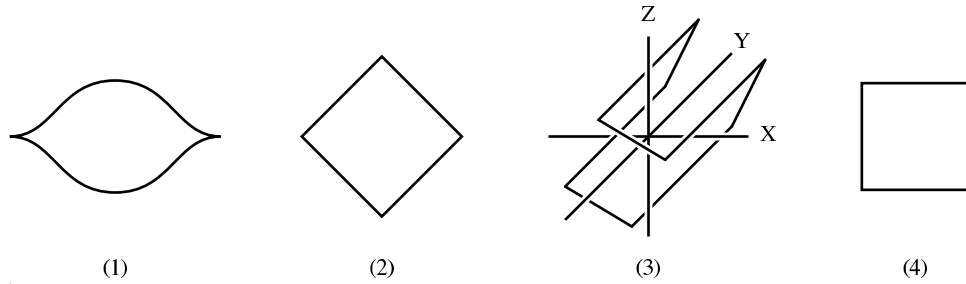


FIGURE 1.

arc in a rectangular diagram to a subarc of the line $\{(x, -1, z) \mid z = -x + v_i\}$ for some $v_i \in \mathbb{R}$, every horizontal arc to a subarc of the line $\{(x, 1, z) \mid z = x + h_j\}$ for some $h_j \in \mathbb{R}$, and every corner to a subarc of the line $\{(x_i, y, z_j) \mid y \in \mathbb{R}\}$ for some $x_i \in \mathbb{R}$ and $z_j \in \mathbb{R}$, we obtain a Legendrian link in $(\mathbb{R}^3, \xi_{\text{std}})$ from a rectangular diagram. We notice that the Legendrian link constructed as above from a rectangular diagram is Legendrian isotopic to the Legendrian link corresponding to the front projection. For example, the diagram in Figure 1 (4) corresponds to a Legendrian knot, illustrated in Figure 1 (3), consisting of the following eight arcs:

$$\begin{aligned} &\{(x, -1, z) \mid x + z = 1, -\varepsilon \leq x \leq 1 - \varepsilon, \varepsilon \leq z \leq 1 + \varepsilon\}, \{(-\varepsilon, y, 1 + \varepsilon) \mid -1 \leq y \leq 1\}, \\ &\{(x, 1, z) \mid x - z = -1, -1 - \varepsilon \leq x \leq -\varepsilon, \varepsilon \leq z \leq 1 + \varepsilon\}, \{(-1 - \varepsilon, y, \varepsilon) \mid -1 \leq y \leq 1\}, \\ &\{(x, -1, z) \mid x + z = -1, -1 - \varepsilon \leq x \leq -\varepsilon, -1 + \varepsilon \leq z \leq \varepsilon\}, \{(-\varepsilon, y, -1 + \varepsilon) \mid -1 \leq y \leq 1\}, \\ &\{(x, 1, z) \mid x - z = 1, -\varepsilon \leq x \leq 1 - \varepsilon, -1 + \varepsilon \leq z \leq \varepsilon\}, \{(1 - \varepsilon, y, \varepsilon) \mid -1 \leq y \leq 1\}, \end{aligned}$$

where ε is a small positive number. The union of these arcs is piecewise Legendrian. We obtain a Legendrian knot by Legendrian-smoothing our edgepath in arbitrarily small neighborhoods around the endpoints of each arc.

In §2, we study a similar correspondence between braided rectangular diagrams and Legendrian links in $(\mathbb{R}^3, \xi_{\text{sym}})$. This leads us to Alexander and Markov Theorems for Legendrian links in $(\mathbb{R}^3, \xi_{\text{sym}})$. This answers Problem 2 in [10], implicitly stated also in [4]. Alexander and Markov Theorems for transverse links in $(\mathbb{R}^3, \xi_{\text{sym}})$ was proved in [1], [10] and [13].

Theorem 1.1. [1] (Alexander Theorem for transverse links in $(\mathbb{R}^3, \xi_{\text{sym}})$)

Any transverse link in $(\mathbb{R}^3, \xi_{\text{sym}})$ is transversely isotopic to a closed braid.

Theorem 1.2. [10], [13] (Markov Theorem for transverse links in $(\mathbb{R}^3, \xi_{\text{sym}})$)

Two closed braids represent the same transverse link if and only if they are related by positive stabilizations and conjugation in the braid group.

In §3, we describe a construction of an explicit example of the Etnyre-Honda pair of Legendrian knots [5], which is announced in [9].

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2. ALEXANDER AND MARKOV THEOREMS FOR LEGENDRIAN LINKS IN $(\mathbb{R}^3, \xi_{\text{sym}})$

Let $\{H_\theta \mid 0 \leq \theta < 2\pi\}$ denote an open book decomposition of \mathbb{R}^3 , that is, $\mathbb{R}^3 \setminus \{z\text{-axis}\}$ is fibered by a collection of half-plane fibers H_θ , where the boundary of H_θ is the z -axis. When we use the cylindrical coordinates (r, θ, z) on \mathbb{R}^3 , a fiber H_{θ_0} is the set $\{(r, \theta, z) \mid \theta = \theta_0\}$. An oriented link X in \mathbb{R}^3 is a *closed n -braid* if $X \subset \mathbb{R}^3 \setminus \{z\text{-axis}\}$ intersects each fiber H_θ transversely and positively in n points. Possibly after a small isotopy of X in $\mathbb{R}^3 \setminus \{z\text{-axis}\}$, we can consider a regular projection $\pi: X \rightarrow C_1$ given by $(r, \theta, z) \mapsto (1, \theta, z)$, where $C_1 = \{(r, \theta, z) \mid r = 1\}$. We may assume that the singularities consist of $\pi(X)$ consists of finitely many transverse double points.

A *horizontal arc*, $h \subset C_1$, is an arc with a parameterization $\{(1, \theta(t), z_0) \mid 0 \leq t \leq 1, \theta(t) \in [\theta_1, \theta_2]\}$, where $|\theta_1 - \theta_2| < 2\pi$. The *horizontal level* of h is a fixed constant z_0 , and the *angular support* of h is the interval $[\theta_1, \theta_2]$. Horizontal arcs inherit a natural orientation from the forward direction of the θ coordinate. A *vertical arc*, $v \subset H_{\theta_0}$, is an arc with a parameterization $\{(r(t), \theta_0, z(t)) \mid 0 \leq t \leq 1, r(0) = r(1) = 1; \text{ and } r(t) > 1, \frac{dz(t)}{dt} \neq 0 \text{ for } t \in (0, 1)\}$, where $r(t)$ and $z(t)$ are \mathbb{R} -valued functions that are continuous on $[0, 1]$ and differential on $(0, 1)$. The *angular level* of v is θ_0 , the *vertical support* of v is the interval $[z(0), z(1)]$ or $[z(1), z(0)]$.

An oriented link $X \subset \mathbb{R}^3$ is an *arc presentation* [8] if $X = h_1 \cup v_1 \cup \cdots \cup h_n \cup v_n$ satisfies the following conditions:

- (1) each h_i , ($i = 1, \dots, n$), is an oriented horizontal arc with its inherited orientation agreeing with the orientation of X ,
- (2) each v_i , ($i = 1, \dots, n$), is a vertical arc,
- (3) the intersection $h_i \cap v_j$ consists of a point of $\partial h_i \cap \partial v_j$ for $i = 1, \dots, n$ and $j \equiv \{i - 1, i\} \pmod{n}$, and $h_i \cap v_j = \emptyset$ if $j \not\equiv \{i - 1, i\} \pmod{n}$,
- (4) the horizontal level of each horizontal arc is distinct, and the angular level of each vertical

arc is distinct,

(5) the orientations of vertical arcs are assigned so as to make the components of X oriented.

A projection of an arc presentation of X onto C_1 with over/under informations is called a *braided rectangular diagram*. A projection $\pi(X)$ without the conditions about orientations in (1) and (5) is called a *rectangular diagram*. It is proved in [3] and [8] that every link in \mathbb{R}^3 has a braided rectangular diagram.

By a contactmorphism $(\mathbb{R}^3, \xi_{\text{std}})$ is equivalent to the symmetric contact structure $(\mathbb{R}^3, \xi_{\text{sym}})$ where $\xi_{\text{sym}} = \ker \alpha_{\text{sym}}$ for $\alpha_{\text{sym}} = dz + xdy - ydx$ (in Euclidean coordinates) or $\alpha_{\text{sym}} = dz + r^2 d\theta$ (in cylindrical coordinates). The symmetric contact structure was the structure utilized by D. Bennequin [1] in his classical argument the any transversal knot is transversally isotopic to a braid—the z -axis being the designated braid axis.

A rectangular diagram on C_1 corresponds to a Legendrian link in $(\mathbb{R}^3, \xi_{\text{sym}})$ as follows. First we notice that the characteristic foliation on the cylinder $\{(r, \theta, z) \mid r = r_0\}$ consists of “spiral” curves of slope $\frac{dz}{d\theta} = -r_0^2$, and that the characteristic foliation on the plane $\{(r, \theta, z) \mid \theta = \theta_0\}$ consists of lines $\{(r, \theta_0, z_0) \mid r > 0\}$ for $z_0 \in \mathbb{R}$. Change every horizontal arc in a rectangular diagram to a subarc, a *near-horizontal arc*, of the characteristic foliation on the cylinder $\{(r, \theta, z) \mid r = r_1\}$ with r_1 sufficiently small, and every vertical arc to a subarc, a *near-vertical arc*, of the characteristic foliation on the cylinder $\{(r, \theta, z) \mid r = r_2\}$ with r_2 sufficiently large. Change every corner in a rectangular diagram to a subarc of the line $\{(r, \theta, z) \mid \theta = \theta_0, r > 0\}$ which connects one endpoint of a near-horizontal arc on the cylinder $\{(r, \theta, z) \mid r = r_1\}$ and one endpoint of a near-vertical arc on the cylinder $\{(r, \theta, z) \mid r = r_2\}$. Adjusting the r -coordinates of near-horizontal and near-vertical arcs properly and Legendrian smoothing in arbitrarily small neighborhoods of the arc endpoints, we obtain a Legendrian link L in $(\mathbb{R}^3, \xi_{\text{sym}})$ from a braided rectangular diagram. The vertical (resp. angular) support of a near-horizontal (resp. near-vertical) arc of L is the interval in the z -coordinate (resp. θ -coordinate) containing the arc. We may Legendrian isotope L so that:

(1) the vertical support of a near-horizontal arc of L is sufficiently small, and is disjoint from each other,

(2) the angular support of a near-vertical arc of L is sufficiently small, and is disjoint from each other.

A Legendrian link L has a *horizontal/vertical disjoint property* if the above conditions are satisfied.

Next we construct a rectangular diagram from a Legendrian link in $(\mathbb{R}^3, \xi_{\text{sym}})$.

Lemma 2.1. *Let L be a Legendrian link in $(\mathbb{R}^3, \xi_{\text{sym}})$. Then L may be Legendrian isotoped to L' so that L' is in the half-space with $y > 0$. In particular, $L' \cap H_\pi = \emptyset$.*

Proof. Instead of Legendrian isotoping L , we describe a contactomorphism of $(\mathbb{R}^3, \xi_{\text{sym}})$ that takes L to L' . Let f be a diffeomorphism of \mathbb{R}^3 defined by $(x, y, z) \mapsto (x+K, y+K, z+K(x-y))$, where $K \in \mathbb{R}$. Starting with the Euclidean version of α_{sym} , we then obtain $(\mathbb{R}^3, \xi'_{\text{sym}})$, where ξ'_{sym} is defined by the kernel of the 1-form $\alpha'_{\text{sym}} = d(z+K(x-y)) + (x+K)d(y+K) - (y+K)d(x+K) = dz + xdy - ydx$. By the compactness of L , for a large enough K the image of L by f , L' , will be in the half-space $y > 0$. In particular, $L' \cap H_\pi = \emptyset$. \square

Let L be a Legendrian link in $(\mathbb{R}^3, \xi_{\text{sym}})$ that is in the half-space having $y > 0$. Then it does not intersect the z -axis and the image of L onto C_1 by $\pi: (r, \theta, z) \mapsto (1, \theta, z)$, $\pi(L)$ is well-defined. We called $\pi(L)$ the *cylinder projection* of L . We notice that when $L \cap H_\pi = \emptyset$ the front projections of Legendrian links in $(\mathbb{R}^3, \xi_{\text{std}})$ and cylinder projections of Legendrian links in $(\mathbb{R}^3, \xi_{\text{sym}})$ are combinatorial equivalent. We can then adapt the proof of the Reidemeister Theorem for Legendrian links in terms of front projections to the setting of cylinder projections to show the following. See the proof of Theorem B in [12].

Proposition 2.2. *Let L_1 and L_2 be Legendrian links in $(\mathbb{R}^3, \xi_{\text{sym}})$ such that each of L_1 and L_2 are in the half-space having $y > 0$. Let D_1 and D_2 be the cylinder projections of L_1 and L_2 , respectively. Legendrian links L_1 and L_2 are Legendrian isotopic in $\mathbb{R}^3 \setminus \{z\text{-axis}\}$ if and only if D_1 and D_2 are related by regular homotopy and a finite sequence of moves that are obtained from the diagrams in Figure 2 by rotating Θ degree counterclockwise with $0 < \Theta < 90$.*

As we obtain a rectangular diagram from a front projection, we obtain a rectangular diagram on C_1 from a cylinder projection by changing every point with $\frac{dz}{d\theta} = -1$ to a corner.

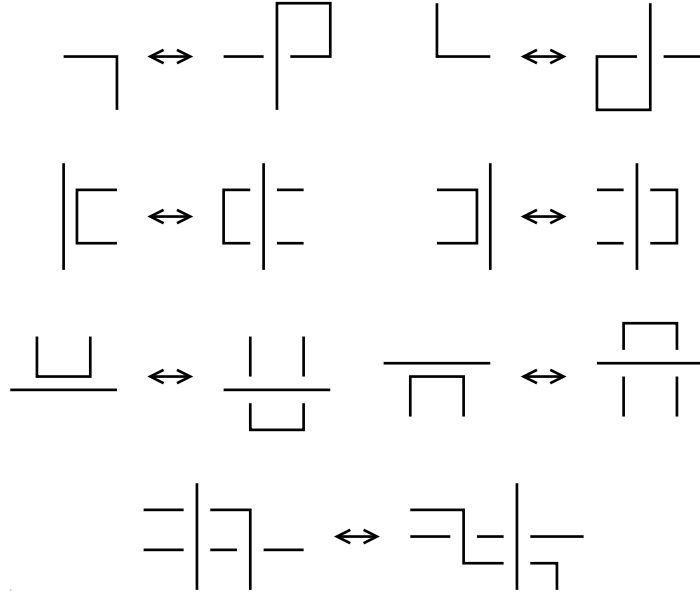


FIGURE 2.

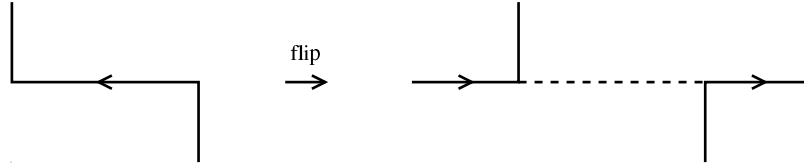


FIGURE 3.

Next we define an operation, called a *flip*. Let $L = \cup_{i=1}^n (h_i \cup c(h_i) \cup v_i \cup c(v_i))$ be a Legendrian link that corresponds to a rectangular diagram, where h_i (resp. v_i) is a near-horizontal (resp. near-vertical) arc, and $c(h_i)$ connects h_i and v_i , and $c(v_i)$ connects v_i and h_{i+1} . We may assume that a Legendrian link L has a horizontal/vertical disjoint property. Moreover, by a Legendrian isotopy that corresponds to scaling the the angle between v_{i-1} and v_i we can assume that the angle between the angular support of h_i is π . We may Legendrian isotope L so that the r -coordinate of h_i goes to 0, and that the θ -coordinates of $c(v_{i-1})$ and $c(h_i)$ remain fixed. Then we have to adjust the lengths of $c(v_{i-1})$ and $c(h_i)$, and the r -coordinates of v_{i-1} and v_i , and the lengths of $c(h_{i-1})$ and $c(v_i)$. After these Legendrian isotopies of L , we may assume that L has a horizontal/vertical disjoint property. We may Legendrian isotope L so that h_i shrinks to a point on the z -axis. Therefore the z -coordinate of $c(v_{i-1})$ and $c(h_i)$ are the same. We may

further Legendrian isotope L to L' so that h_i passes through the z -axis to h'_i , that (the angular support of h_i) \cap (the angular support of h'_i) = (the θ -coordinates of $c(v_{i-1})$ and $c(h_i)$), where h'_i is a subarc of L' . By our assumption that the angular support of h_i is π this isotopy will be Legendrian as L passes through the z -axis. This isotopy from L to L' is called a *flip*, and the above argument shows that a flip is a Legendrian isotopy from L to L' .

Theorem 2.3. (Alexander Theorem for Legendrian links in $(\mathbb{R}^3, \xi_{\text{sym}})$)

Every Legendrian link in $(\mathbb{R}^3, \xi_{\text{sym}})$ is Legendrian isotopic to a Legendrian link constructed from a braided rectangular diagram.

Proof. Let D be a rectangular diagram of a Legendrian link L with a horizontal/vertical disjoint property. Let h be a near-horizontal arc of L such that the induced orientation of h from that of L disagrees with the forward direction of the θ -coordinate. We apply a flip to every such near-horizontal arc h . Then L corresponds to a braided rectangular diagram. \square

When a Legendrian link L intersects the z -axis during Legendrian isotopy of L in $(\mathbb{R}^3, \xi_{\text{sym}})$, a neighborhood of the intersection of L with the z -axis is horizontal. Therefore we may assume that their rectangular diagrams are related by one flip, so we have the following.

Theorem 2.4. (Reidemeister Theorem for Legendrian links in terms of rectangular diagrams)

Let L_1 and L_2 be Legendrian links in $(\mathbb{R}^3, \xi_{\text{sym}})$, and let D_1 and D_2 be the rectangular diagrams of L_1 and L_2 , respectively. Legendrian links L_1 and L_2 are Legendrian isotopic in $(\mathbb{R}^3, \xi_{\text{sym}})$ if and only if D_1 and D_2 are related by a finite sequence of flips and moves in Figure 2 on C_1 .

Next we see how positive and negative transverse push-offs are obtained from a Legendrian link corresponding to a braided rectangular diagram. Let $L = \cup_{i=1}^n (h_i \cup c(h_i) \cup v_i \cup c(v_i))$ be a Legendrian link in $(\mathbb{R}^3, \xi_{\text{sym}})$, where h_i (resp. v_i) denotes a near-horizontal (resp. near-vertical) arc. Let α be a subarc of L that is contained in a cylinder $C_{r_1} = \{(r, \theta, z) \mid r = r_1\}$, so α is h_i or v_i . Let $[\theta_1, \theta_2]$ be the angular support of α . Let $z(\theta)$ be the z -coordinate at θ of the line in the characteristic foliation on C_{r_1} containing α , defined in the θ -interval $[\theta_1 - \varepsilon, \theta_2 + \varepsilon]$, where ε is a small positive number. Let $\delta(\alpha)$ be a neighborhood of α on C_{r_1} described as $\delta(\alpha) = \{(r_1, \theta, z) \mid \theta_1 - \varepsilon \leq \theta \leq \theta_2 + \varepsilon, z(\theta) - \varepsilon \leq z \leq z(\theta) + \varepsilon\}$. Let $\Delta(\alpha)$ be a neighborhood

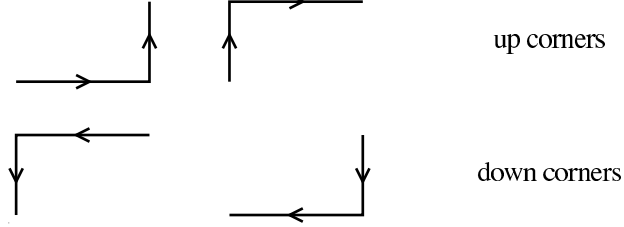


FIGURE 4.

of α in \mathbb{R}^3 described as $\Delta(\alpha) = [r_1 - \varepsilon, r_1 + \varepsilon] \times \delta(\alpha) = \{(r, \theta, z) \mid r_1 - \varepsilon \leq r \leq r_1 + \varepsilon, \theta_1 - \varepsilon \leq \theta \leq \theta_2 + \varepsilon, z(\theta) - \varepsilon \leq z \leq z(\theta) + \varepsilon\}$.

Let β be a subarc of L which is contained in a line $\ell(\theta_1, z_1) = \{(r, \theta, z) \mid \theta = \theta_1, z = z_1\}$, that is, $\beta = \{(r, \theta_1, z_1) \mid r_1 \leq r \leq r_2\}$, so β is $c(h_i)$ or $c(v_i)$. Let $\beta_\varepsilon = \{(r, \theta_1, z_1) \mid r_1 - \varepsilon \leq r \leq r_2 + \varepsilon\}$ be a neighborhood of β in $\ell(\theta_1, z_1)$. Let $\delta(\beta_\varepsilon, r)$ be a neighborhood of $\beta_\varepsilon \cap C_r$ on C_r described as $\delta(\beta_\varepsilon, r_0) = \{(r_0, \theta, z) \mid \theta_1 - \varepsilon \leq \theta \leq \theta_1 + \varepsilon, z(\theta) - \varepsilon \leq z \leq z(\theta) + \varepsilon\}$, where $z(\theta)$ denotes the z -coordinate of the integral line in the characteristic foliation on C_{r_0} containing the point $\beta_\varepsilon \cap C_{r_0}$, defined in the θ -interval $[\theta_1 - \varepsilon, \theta_2 + \varepsilon]$. Let $\Delta(\beta)$ be a neighborhood of β in \mathbb{R}^3 described as $\Delta(\beta) = [r_1 - \varepsilon, r_2 + \varepsilon] \times \delta(\beta_\varepsilon, r) = \{(r, \theta, z) \mid r_1 - \varepsilon \leq r \leq r_2 + \varepsilon, \theta_1 - \varepsilon \leq \theta \leq \theta_1 + \varepsilon, z(\theta) - \varepsilon \leq z \leq z(\theta) + \varepsilon\}$.

Let $\Delta(L)$ be a neighborhood of L constructed as $\Delta(L) = \cup_{i=1}^{2n} (\Delta(\alpha_i) \cup \Delta(\beta_i))$, where $L = \cup_{i=1}^{2n} (\alpha_i \cup \beta_i)$. After a small isotopy of $\Delta(L)$, we may assume that $\partial\Delta(L)$ consists of four sides. Two of the four sides of $\partial\Delta(L)$ each contain one Legendrian divide. Let $T_+(L)$ and $T_-(L)$ be the center lines of the other two sides. Then $T_+(L)$ and $T_-(L)$ are transverse push-offs of L in $(\mathbb{R}^3, \xi_{\text{sym}})$. We note that $T_+(L)$ (resp. $T_-(L)$) is a positive (resp. negative) transverse push-off of L .

Lemma 2.5. *Let L be a Legendrian link in $(\mathbb{R}^3, \xi_{\text{sym}})$ represented as a rectangular diagram D . Then we have $tb(L) = \omega(D) - \frac{1}{2}(d(D) + u(D))$ and $r(L) = n(D) + \frac{1}{2}(d(D) - u(D))$, where $u(D)$ (resp. $d(D)$) denotes the number of up (resp. down) corners of D , illustrated in Figure 4, and $n(D)$ is the algebraic winding number of L around the z -axis, and $\omega(D)$ is the algebraic crossing number of D on C_1 .*

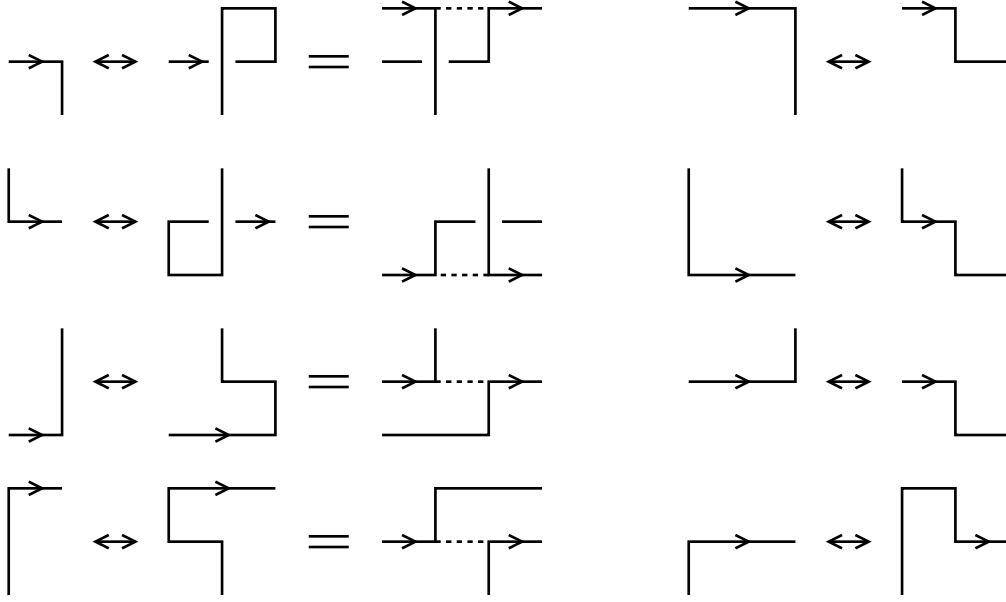


FIGURE 5.

Proof. Let $v = \frac{\partial}{\partial z}$ be a vector field on \mathbb{R}^3 . For any Legendrian knot L , v is a vector field transverse to ξ_{sym} along L . So we have $tb(L) = \omega(D) - \frac{1}{2}(d(D) + u(D))$.

Let $w = \frac{\partial}{\partial r}$ be a vector field on $\{x \geq \varepsilon\}$, $w = -\frac{\partial}{\partial r}$ a vector field on $\{x \leq -\varepsilon\}$. We define a vector field w on $\{-\varepsilon < x < \varepsilon\}$ by interpolating between these two choices by rotating clockwise in the contact planes. We may Legendrian isotope L so that all the vertical arcs of D are contained in $\{x \geq \varepsilon\}$. Then we have $r(L) = \frac{1}{2}(2n(D) + d(D) - u(D)) = n(D) + \frac{1}{2}(d(D) - u(D))$. \square

Similar arguments as above prove the following.

Lemma 2.6. *Let $T_+(L)$ be a positive transverse push-off of a Legendrian knot L in $(\mathbb{R}^3, \xi_{\text{sym}})$ corresponding to a braided rectangular diagram D . Then we have $sl(T_+(L)) = \omega(D) - n(D)$.*

Proposition 4 in [3] and Theorem 2.4 prove the following.

Theorem 2.7. (Markov Theorem for Legendrian links in $(\mathbb{R}^3, \xi_{\text{sym}})$)

Let D_1 and D_2 be braided rectangular diagrams on C_1 , and let L_1 and L_2 be Legendrian links corresponding to D_1 and D_2 , respectively. Two Legendrian links L_1 and L_2 are Legendrian

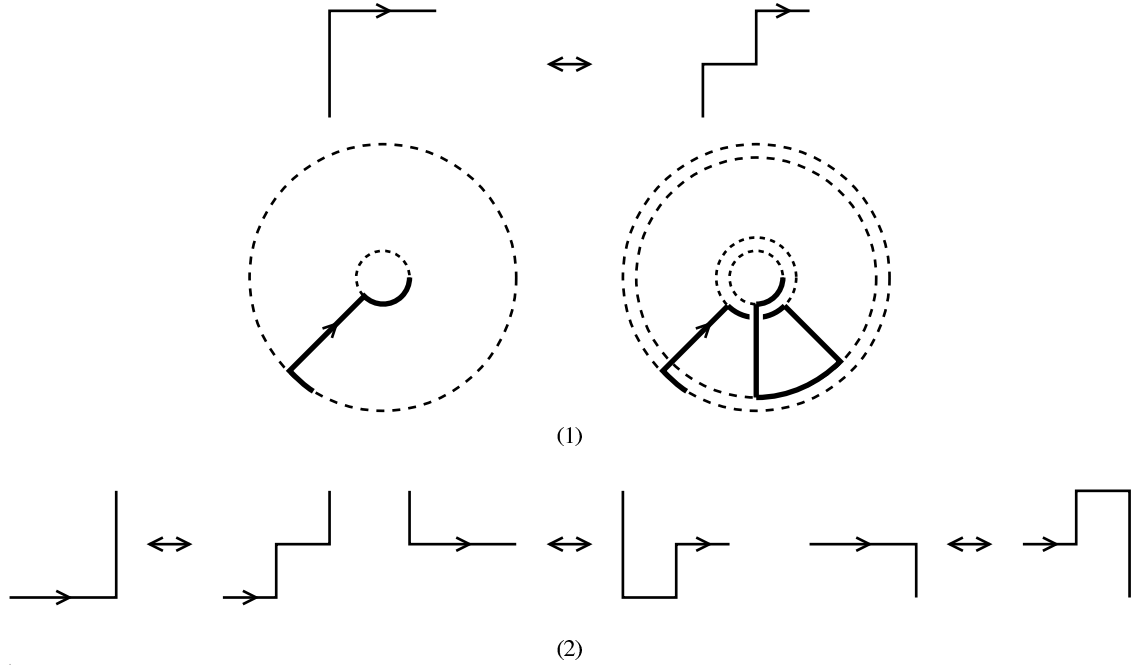


FIGURE 6.

isotopic in $(\mathbb{R}^3, \xi_{\text{sym}})$ if and only if D_1 is obtained from D_2 by a finite sequence of moves illustrated in Figure 5.

Remark 2.8. Let $p: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a projection onto the xy -plane, and L be a Legendrian link in $(\mathbb{R}^3, \xi_{\text{sym}})$. We denote $p(L)$ with over/under information by $p'(L)$. Let $D_1 \rightarrow D_2$ be a move in Figure 6. We note that $p'(L_1) \rightarrow p'(L_2)$ is a negative Reidemeister move of type I in knot theory, where L_i ($i = 1, 2$) is a Legendrian link corresponding to a rectangular diagram D_i . See Figure 6 (1). This may be seen as a negative “local-(de)stabilization”. We note also that $T_+(L_1) \rightarrow T_+(L_2)$ is a transverse isotopy. See Theorem 1.2.

Remark 2.9. Let $D_1 \rightarrow D_2$ be a move in Figure 7, and L_i ($i = 1, 2$) be a Legendrian link corresponding to a rectangular diagram D_i . Then $T_+(L_1) \rightarrow T_+(L_2)$ is a negative (de)stabilization as closed braids, so Theorem 1.2 shows that $T_+(L_1) \rightarrow T_+(L_2)$ is not a transverse isotopy.

Similar results as Theorems 2.4 and 2.7 are obtained in [11] in terms of rectangular diagrams, also known as grid-link diagrams.

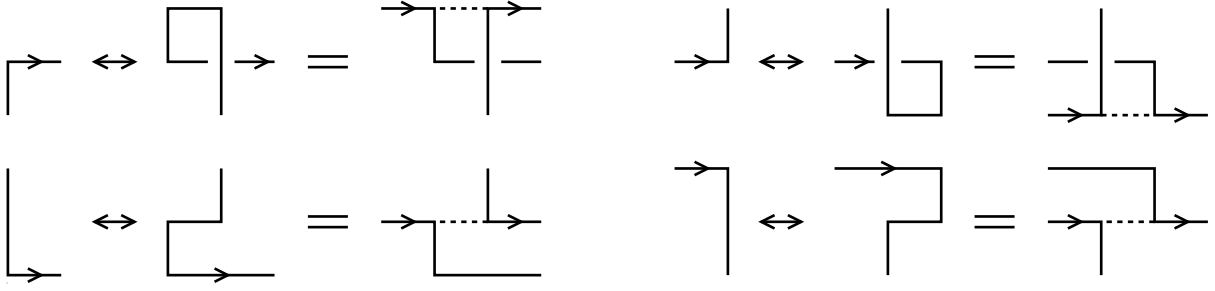


FIGURE 7.

3. CONSTRUCTION OF ETNYRE-HONDA PAIR

We start with a standardly embedded torus U in \mathbb{R}^3 . This torus U may be described as $\{(r, \theta, z) \mid (r-2)^2 + z^2 = 1\}$ in the cylindrical coordinate of \mathbb{R}^3 . Choose two sets of $pq + p + q$ numbers z_1, \dots, z_{pq+p+q} and $\theta_1, \dots, \theta_{pq+p+q}$ satisfying the conditions $-1 < z_1 < \dots < z_{pq+p+q} < 1$ and $0 < \theta_1 < \dots < \theta_{pq+p+q} < \pi$. The intersection of U with the plane $\{(r, \theta, z) \mid z = z_i\}$ consists of two circles L^i and ℓ^i , where the r -coordinate of L^i is larger than that of ℓ^i . Let M^i be the intersection of U with the plane $\{(r, \theta, z) \mid \theta = \theta_i\}$, which is a circle. The union of circles $\cup_{i=1}^{pq+p+q} (L^i \cup M^i)$ separate U into $(pq + p + q)^2$ squares. Choose one rectangle W_1 on U with $z_{pq+q} \leq z \leq z_{pq+p+q}$ and $\theta_1 \leq \theta \leq \theta_{q+1}$ occupying pq squares. See Figure 8 (2). Choose a rectangle W_{k+1} on U occupying pq squares in p rows and q columns so that the upper right corner of W_k is identified with the lower left corner of W_{k+1} for $k = 1, \dots, pq + p + q$. Isotope W_k so that $W_k \cap \{(r, \theta, z) \mid (r-2)^2 + z^2 \leq 1\}$ consists of a subarc of each of L^i and L^{i+p} . See Figure 8 (1).

Let Z^i be the subdisc of the plane $\{(r, \theta, z) \mid z = z_i\}$ with $\partial Z^i = L^i$. Let \mathcal{T} denote a torus $\partial N_\varepsilon((\cup_{i=1}^{pq+p+q} Z^i) \cup (\cup_{j=1}^{pq+p+q} W_j); S^3)$, that is the boundary of an ε -neighborhood of the branched surface $(\cup_{i=1}^{pq+p+q} Z^i) \cup (\cup_{j=1}^{pq+p+q} W_j)$ in S^3 . Figure 9 illustrates a tiling obtained from a braid foliation $\mathcal{T} \cap \{H(\theta)\}$ on \mathcal{T} , where the point with the mark $+k$ (resp. $-k$) represents the intersection of $\partial N(Z^k)$ and the z -axis with larger (resp. smaller) z -coordinate, and the point with the mark $k+$ (resp. $k-$) represents the hyperbolic singularity on the plane $\{(r, \theta, z) \mid \theta = \theta_i + \varepsilon\}$ (resp. $\{(r, \theta, z) \mid \theta = \theta_i - \varepsilon\}$) corresponding to $\partial N(W_j)$ (resp. $\partial N(W_{j+1})$). We notice that each of G_{++} and G_{--} consists of a circle, where G_{++} and G_{--} are the graphs defined in

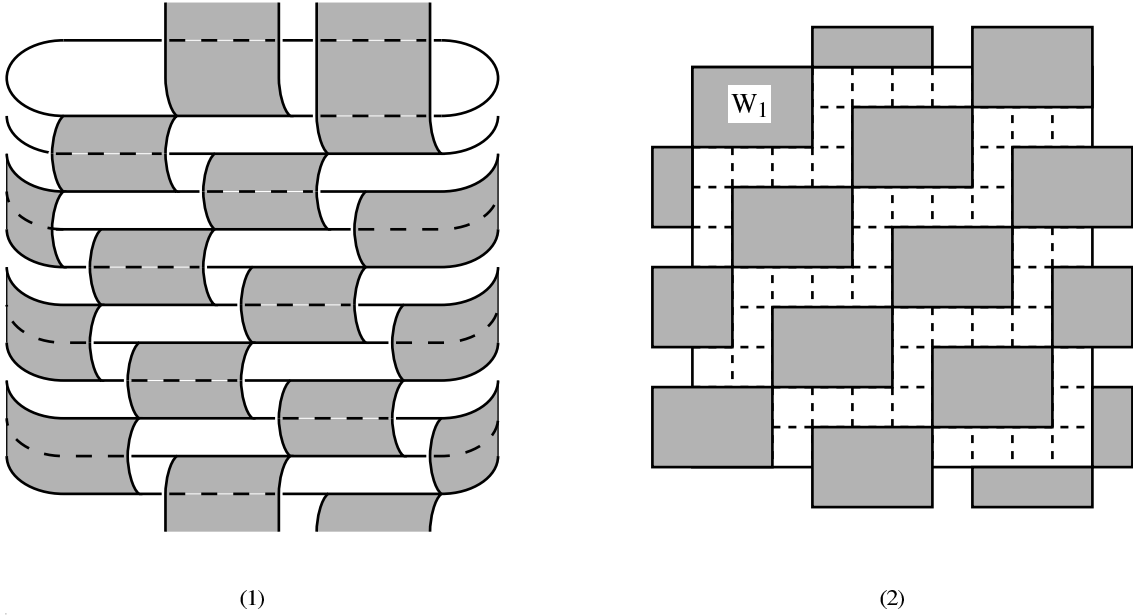


FIGURE 8.

[2]. Let C_1 and C_2 be the annuli $\mathcal{T} \setminus (G_{++} \cup G_{--})$, and let c_1 and c_2 be the core curve of C_1 and C_2 , respectively. When $p = 2$ and $q = 3$, c_1 is a curve of slope $-\frac{2}{11}$ on \mathcal{T} with respect to the coordinate system \mathcal{C}' in [5], where the boundary of the cabling annulus has slope $\frac{1}{0}$, and the meridian of \mathcal{T} has slope $\frac{0}{1}$. Figure 9 illustrates c_1 on \mathcal{T} .

Next we look at \mathcal{T} in $(\mathbb{R}^3, \xi_{\text{sym}})$. Isotope \mathcal{T} so that L^i has sufficiently small r coordinate, and that the hyperbolic singularity on $\partial N(W_i)$ has sufficiently large r coordinate. Each elliptic singularity in the characteristic foliation on \mathcal{T} corresponds to an intersection of \mathcal{T} with the z -axis, which is an elliptic singularity in a braid foliation. Each hyperbolic singularity in the characteristic foliation on \mathcal{T} corresponds to a hyperbolic singularity on \mathcal{T} in a braid foliation. Thus the characteristic foliation on \mathcal{T} is isotopic to the corresponding braid foliation on \mathcal{T} .

We may use the Giroux Elimination Lemma[6] to isotope \mathcal{T} in a small neighborhood of \mathcal{T} , and we may eliminate G_{++} and G_{--} . Then \mathcal{T} is a convex torus, and c_1 and c_2 are Legendrian divides on \mathcal{T} with slope $-\frac{2}{11}$ with respect to \mathcal{C}' . Figure 10 illustrates a train-track on \mathcal{T} constructed from c_1 , c_2 and a small arc connecting them. Let $\ell(r, s)$ denote a simple closed curve supported by the train-track with weights r , s and $r + s$, as illustrated in the figure.

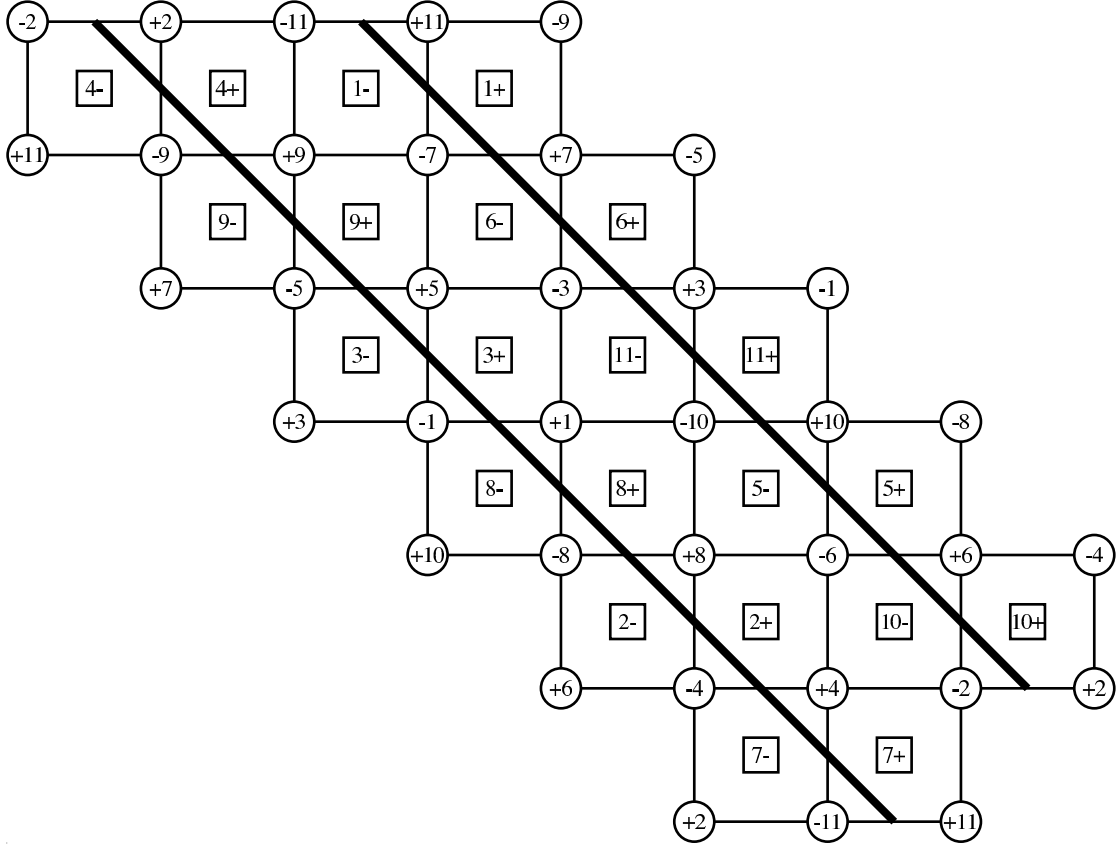


FIGURE 9.

The topological knot type of $\ell(r, s)$ is a $(2r + s, r + s)$ -cable of a $(2, 3)$ -torus knot. the Giroux Flexibility Theorem[6] allows one to isotope \mathcal{T} in a small neighborhood of \mathcal{T} so that $\ell(r, s)$ is a Legendrian ruling curve on \mathcal{T} with slope $-\frac{2r+s}{11r+5s}$ with respect to \mathcal{C}' on \mathcal{T} . This Legendrian ruling curve may correspond to a braided rectangular diagram. When $r = s = 1$, $\ell(1, 1)$ is L_+ in [5]. A braided rectangular diagram of $\ell(1, 1)$ is obtained from the diagram in Figure 11. It is easy to construct K_+ in [5], as illustrated in Figure 11. A similar proof as in Lemma 6.3 in [5] shows that $\ell(r, s)$ does not admit a Legendrian destabilization.

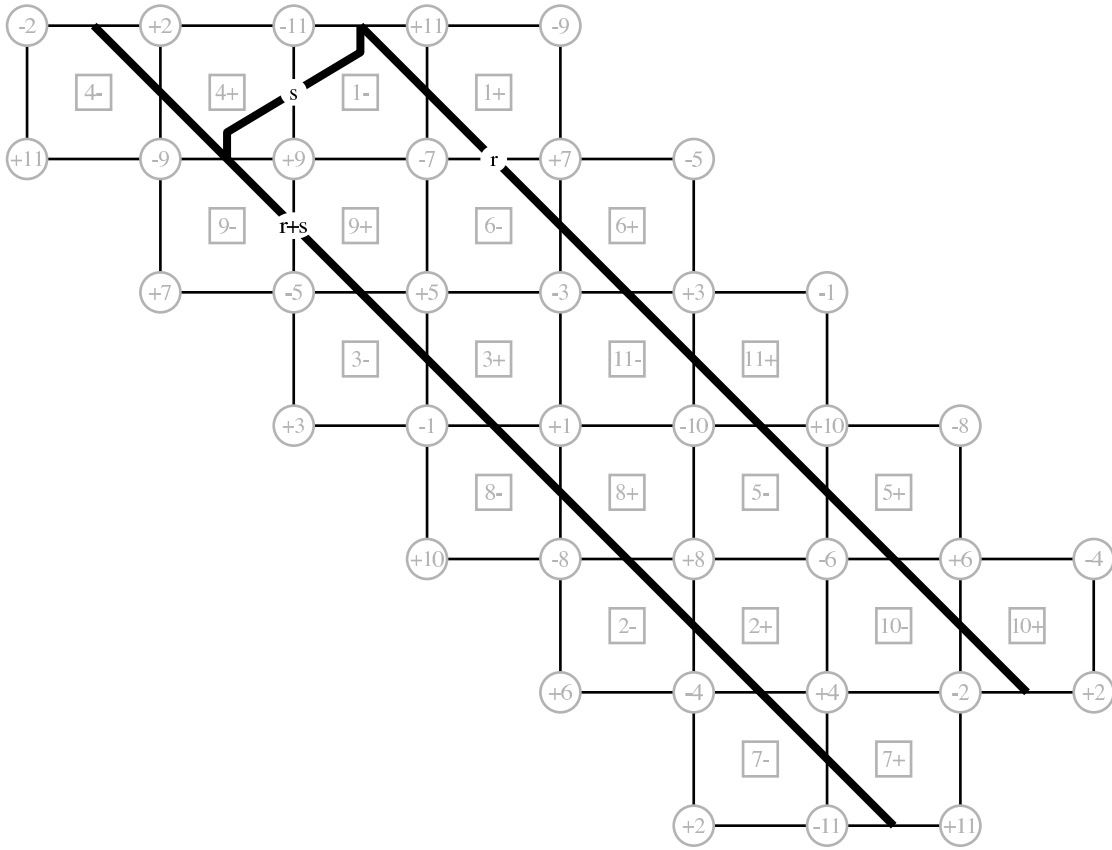


FIGURE 10.

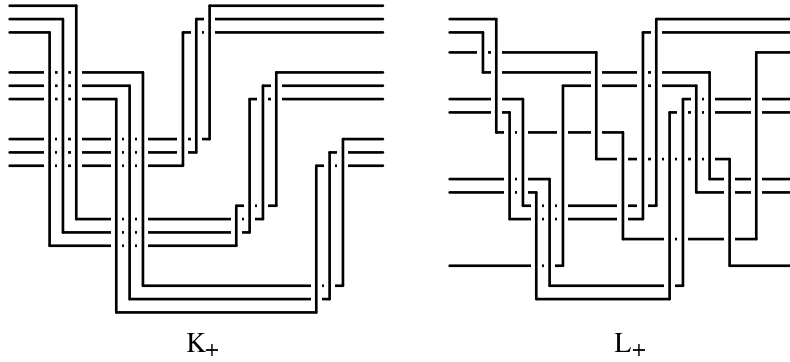


FIGURE 11.

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